A DEGREE INEQUALITY FOR LIE ALGEBRAS WITH A REGULAR POISSON SEMI-CENTER

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Abstract. For Lie algebras whose Poisson semi-center is a polynomial ring we give a bound for the sum of the degrees of the generating semi-invariants. This bound was previously known in many special cases.

1. Introduction

In this paper we work over an algebraically closed base field k of characteristic zero. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. A non-zero element $f \in S\mathfrak{g}$ is called a *semi-invariant with weight* $\chi \in \mathfrak{g}^*$ if for all $v \in \mathfrak{g}$ we have

$$\operatorname{ad}(v)(f) = \chi(v)f$$

We say that a semi-invariant is *proper* if $\chi \neq 0$. The k-algebra generated by the semi-invariants in $S\mathfrak{g}$ is denoted by $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$. This ring is called the *Poisson semi-center* of $S\mathfrak{g}$.

The *stabilizer* of $x \in \mathfrak{g}^*$ is denoted by $\mathfrak{g}^*_x$. I.e. $\mathfrak{g}_x = \{v \in \mathfrak{g} \mid \forall w \in \mathfrak{g} : x([v,w]) = 0\}$. The minimal value of $\dim \mathfrak{g}_x$ is called the *index* of $\mathfrak{g}$ and is denoted by $i(\mathfrak{g})$. An element $x \in \mathfrak{g}^*$ is called *regular* if $\dim \mathfrak{g}^*_x = i(\mathfrak{g})$. The regular elements form an open dense subset of $\mathfrak{g}^*$ which we denote by $\mathfrak{g}^*_{\mathrm{reg}}$.

The following is our main result.

Theorem 1.1. *(see Prop. 3.1 and Prop. 5.7 below.) Assume that $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$ is freely generated by homogeneous elements $f_1, \dots, f_r$. Then*

$$\sum_{i=1}^{r} \deg f_i \leq \frac{1}{2}\bigl(\dim \mathfrak{g} + i(\mathfrak{g})\bigr) \tag{1.1}$$

It is well-known that (1.1) holds for semi-simple Lie algebras [5, Thm. 7.3.8] and Frobenius Lie algebras [2, pp. 339–343]. Numerous other special cases are known (e.g. [7, 14]).

For Theorem 1.1 to be valid in the stated generality it is essential that we consider semi-invariants instead of invariants, as the following trivial example by Panyushev shows.

Example 1.2. Let $\mathfrak{g} = kv_1 + kv_2 + kv_3 + kv_4$ with non-trivial brackets $[v_1, v_2] = v_2$, $[v_1, v_3] = v_3$, $[v_1, v_4] = -v_4$. Then $\dim \mathfrak{g} = 4$, $i(\mathfrak{g}) = 2$. The generating invariants are v_2v_4 and v_3v_4. So the sum of their degrees is $2 + 2 = 4$ which is strictly bigger than $1/2(\dim \mathfrak{g} + i(\mathfrak{g})) = 3$. However the generating *semi-invariants* are v_2, v_3, v_4 and the sum of their degrees is 3, which does not violate the inequality.

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For brevity we will call a Lie algebra *coregular* if $(S\mathfrak{g})^{\mathfrak{g}}$ is a polynomial ring.

Corollary 1.3. *Assume that* $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$ *and* $\mathfrak{g}$ *is coregular with center* $Z(\mathfrak{g})$*. Then*

$$3\,i(\mathfrak{g}) \leq \dim \mathfrak{g} + 2\dim Z(\mathfrak{g}) \tag{1.2}$$

Proof. In this situation we have the equality $r = i(\mathfrak{g})$ (see Proposition 4.1 below). The observation that $\deg f_i \geq 2$, unless $f_i \in Z(\mathfrak{g})$ yields

$$\dim Z(\mathfrak{g}) + 2(i(\mathfrak{g}) - \dim Z(\mathfrak{g})) \leq \sum_{i=1}^{r} \deg f_i \leq \frac{1}{2}(\dim \mathfrak{g} + i(\mathfrak{g}))$$

which translates into (1.2). □

The number on the right hand side of (1.1) occurs frequently in the theory of enveloping algebras. For example it is an upper bound for the transcendence degree of a maximal commutative subfield of the division ring of fractions of $U\mathfrak{g}$ and this bound can be achieved in many cases [11, 16]. Likewise by a result of Sadetov [21] it is the maximum transcendence degree of a Poisson commutative subfield of the field of fractions of $S\mathfrak{g}$.

For the proof of Theorem 1.1 we first reduce to the case that there are no proper semi-invariants (i.e. $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$). In this situation one may prove a result which is more precise than Theorem 1.1. Assume first that $\mathfrak{g}$ is non-abelian. Let $B = ([v_i, v_j])_{ij} \in M_n(S\mathfrak{g})$ be the *structure matrix* of $\mathfrak{g}$ where $v_1, \ldots, v_n$ is an arbitrary basis of $\mathfrak{g}$. Put $s = \dim \mathfrak{g} - i(\mathfrak{g})$. Then the greatest common divisor of the $s \times s$-minors in B is a semi-invariant in $S\mathfrak{g}$ [2]. Below we will call it the *fundamental semi-invariant* and we denote its degree by $d(\mathfrak{g})$. If $\mathfrak{g}$ is abelian we put $d(\mathfrak{g}) = 0$.

Proposition 1.4. *(see Prop. 5.7.) Assume that* $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$ *and* $\mathfrak{g}$ *is coregular. Then we have*

$$\sum_{i=1}^{r} \deg f_i = \frac{1}{2}\big(\dim \mathfrak{g} + i(\mathfrak{g}) - d(\mathfrak{g})\big) \tag{1.3}$$

Taking into account Propositions 4.1 and 5.1 below, this result may also be deduced from [17, Remark 1.6.3]. Our proof uses the general techniques from [12, 13] and is quite different from [17]. We obtain a certain nice complex of length three, consisting of free $S\mathfrak{g}$-modules which, besides implying (1.3), yields some additional information on $\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{reg}}$ (see Proposition 1.6 below).

Corollary 1.5. *Assume that* $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$ *and* $\mathfrak{g}$ *is coregular. Then* (1.1) *is an equality if and only if* $\mathfrak{g}^* \setminus \mathfrak{g}^*_{reg}$ *has codimension* ≥ 2*.*

This follows from the easily verified fact that $d(\mathfrak{g}) = 0$ if and only if $\operatorname{codim}_{\mathfrak{g}^*}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{reg}}) \geq 2$.

Proposition 1.4 is false without the assumption $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$. Counter examples are given by Frobenius Lie algebras. By definition these satisfy $i(\mathfrak{g}) = 0$ and thus the fundamental semi-invariant is equal to $\det B$. Hence $d(\mathfrak{g}) = \dim \mathfrak{g}$ and the righthand side of (1.3) is zero. Since $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$ is freely generated by the irreducible factors of $\det B$ [2] the lefthand side of (1.3) is never zero. It would be interesting to find a version of Proposition 1.4 which holds in the same generality as Theorem 1.1.

As mentioned above we may use our methods to obtain some additional necessary conditions for coregularity. As $\mathfrak{g}$ acts by derivations on $S\mathfrak{g}$ we have a $S\mathfrak{g}$-linear map

$$\rho : S\mathfrak{g} \otimes \mathfrak{g} \to \mathrm{Der}_k(S\mathfrak{g}) = S\mathfrak{g} \otimes \mathfrak{g}^*$$

Proposition 1.6. *(see Prop. 5.4,5.10) Assume that* $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$ *and* $\mathfrak{g}$ *is coregular. Then*

1. $\ker \rho$ *is a free* $S\mathfrak{g}$*-module;*
2. *if* $\mathfrak{g}$ *is not abelian then* $\operatorname{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{reg}) \leq 3$*.*
3. *If* $\operatorname{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{reg}) = 3$ *then* $\mathfrak{g}^* \setminus \mathfrak{g}^*_{reg}$ *is purely of codimension three.*

Example 1.7. We illustrate the above results with an easy example. For $n \geq 3$ let $\mathfrak{g} = L(n)$ be the n-dimensional *standard filiform Lie algebra*. $L(n)$ has a basis $v_1, \ldots, v_n$ and non-trivial brackets $[v_1, v_i] = v_{i+1}$ for $i = 2, \ldots, n-1$. In this case

$$(S\mathfrak{g})^{\mathfrak{g}} = k[v_2, \ldots, v_n]^e$$

where e is the derivation

$$e = \sum_{i=2}^{n-1} v_{i+1} \frac{\partial}{\partial v_i}$$

Dixmier verified by direct computation that $L(3)$, $L(4)$ are coregular but $L(5)$ is not [4]. From the classical correspondence between $\mathbb{G}_a$-invariants and SL_2-covariants (e.g. [8, §33]) one obtains that $L(n)$ is coregular if and only if $n < 5$ (see e.g. [22]).

In order to apply the criteria given above it is advantageous to use the structure matrix B which was already introduced. It is easy to see that the $S\mathfrak{g}$-linear map ρ is represented by the matrix B. Furthermore $\mathfrak{g}_x = \ker x(B)$. If we write $r(\mathfrak{g})$ for the rank of B over the quotient field of $S\mathfrak{g}$ then $i(\mathfrak{g}) = \dim \mathfrak{g} - r(\mathfrak{g})$.

In the case of $L(n)$ the structure matrix looks like

$$\begin{pmatrix} 0 & v_3 & \cdots & v_n & 0 \\ -v_3 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ -v_n & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

We deduce $i(\mathfrak{g}) = n - 2$. Furthermore the fundamental semi-invariant is 1 unless $n = 3$ in which case it is v_3^2. Thus

$$d(\mathfrak{g}) = \begin{cases} 0 & \text{if } n > 3 \\ 2 & \text{if } n = 3 \end{cases}$$

Since $\mathfrak{g}$ is nilpotent there are no proper semi-invariants. As $Z(\mathfrak{g}) = kv_n$ the numerical criterion (1.2) for coregularity becomes

$$3(n-2) \leq n + 2$$

which holds iff $n \leq 4$. Hence the non-coregularity of $L(n)$ for $n \geq 5$ is detected by (1.2).

We have

$$\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{reg}} = \{x \in \mathfrak{g}^* \mid x(v_i) = 0 \text{ for } i = 3, \ldots, n\}$$

Thus $\operatorname{codim}(\mathfrak{g}^* \setminus \mathfrak{g}^*_{\mathrm{reg}}) = n - 2$ and so the fact that $L(n)$ is not coregular for $n \geq 6$ is detected by the numerical criterion Prop. 1.6(2).

If $n = 5$ then

$$\ker \rho = \{(A_1, \ldots, A_5) \in k[v_1, \ldots, v_5] \mid A_2 v_3 + A_3 v_4 + A_4 v_5 = 0, A_1 v_3 = A_1 v_4 = A_1 v_5 = 0\}$$

This kernel is minimally generated by $w_1 = (0, v_4, -v_3, 0, 0)$, $w_2 = (0, 0, v_5, -v_4, 0)$, $w_3 = (0, v_5, 0, -v_3, 0)$ and $w_4 = (0, 0, 0, 0, 1)$. These generators are related by $v_5 w_1 + v_3 w_2 - v_4 w_3 = 0$ so they are not free. Thus the non-coregularity of $L(5)$ is detected by Prop. 1.6(1) but not by 1.6(2).

Let us now consider $n = 3$. In this case the generating invariant is v_3 and the equality (1.3) becomes $1 = (1/2)(3 + 1 - 2)$.

Assume $n = 4$. Now the generating invariants are v_4 and $v_2 v_4 - (1/2)v_3^2$. Then (1.3) becomes $1 + 2 = (1/2)(4 + 2 - 0)$.

Although not directly related to the content of this paper let us remind the reader that much is known classically about the invariant theory of SL_2. This may be translated back into results about $L(n)$. For $\mathfrak{g} = L(7)$ one finds that $(S\mathfrak{g})^{\mathfrak{g}}$ is minimally generated by 23 elements (see [8, §115]). On the other hand the transcendence degree of the fraction field of $(S\mathfrak{g})^{\mathfrak{g}}$ is only 5.

We refer to [14] for explicit generators of $(S\mathfrak{g})^{\mathfrak{g}}$ for many nilpotent Lie algebras of dimension at most 7.

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2. Preliminaries

Throughout $\mathfrak{g}$ is a finite dimensional Lie algebra. If V is a finite dimensional representation of $\mathfrak{g}$ then we denote by $(SV)^{\mathfrak{g}}_{\mathrm{si}}$ the ring of semi-invariants in SV. Note that if f is a semi-invariant and $g \in SV$ divides f then g is a semi-invariant as well. Thus any semi-invariant in SV is a product of semi-invariants which are irreducible in SV.

If $x \in V^*$ then ∂_x is the derivation of SV such that for $v \in V$ we have $\partial_x(v) = x(v)$.

We equip $S\mathfrak{g}$ with the Kostant-Kirillov Poisson bracket of degree -1

$$\{v_1, v_2\} = [v_1, v_2] \qquad (v_1, v_2 \in \mathfrak{g})$$

If $g \in S\mathfrak{g}$ is a semi-invariant with weight χ then for all $f \in S\mathfrak{g}$ we have $\{f, g\} = \partial_\chi(f) g$. From this we easily deduce the well-known fact that semi-invariants in $S\mathfrak{g}$ Poisson commute.

It will be convenient to introduce the *Lie-Rinehart algebra* $SV \otimes \mathfrak{g}$ [19]. This is a Lie algebra with Lie bracket

$$[f \otimes v, g \otimes w] = f v(g) \otimes w - g w(f) \otimes v + fg \otimes [v, w]$$

Sending $f \otimes v$ to $fv(-)$ defines an $S\mathfrak{g}$-linear Lie algebra homomorphism

$$\rho : SV \otimes \mathfrak{g} \to \mathrm{Der}_k(SV)$$

which is called the *anchor map*. If $(v_i)_i$ is a basis for $\mathfrak{g}$ then the kernel of the anchor map is given by the sums $\sum_i c_i \otimes v_i$ such that $\sum_i c_i v_i(w) = 0$ for all $w \in V$. Note that this kernel is a Lie ideal (as is any kernel of a homomorphism between Lie algebras).

If we use the identification $\mathrm{Der}_k(SV) = SV \otimes V^*$ and we choose bases $(v_i)_{i=1}^n$, $(w_j)_{j=1}^m$ for $\mathfrak{g}$ and V then the anchor map is represented with respect to these bases by the *structure matrix* $(v_i(w_j))_{ji} \in M_{m\times n}(SV)$ of V.

For convenience we will write $r(V)$ for the rank of the structure matrix of V over the field of fractions of SV. If $\mathfrak{g}$ is in doubt we write $r_{\mathfrak{g}}(V)$. There is an open subset V^*_{reg} of V^* such that $x \in V^*_{\mathrm{reg}}$ iff $\dim \mathfrak{g}_x = \dim V - r(V)$ where $\mathfrak{g}_x$ denotes the stabilizer of x. I.e. $\mathfrak{g}_x = \{v \in \mathfrak{g} \mid \forall w \in V : x([v,w]) = 0\}$.

The *fundamental semi-invariant* in SV is defined as the greatest common divisor of the $r(V)\times r(V)$ minors in the structure matrix of SV [2], assuming $r(V) > 0$. If $C \subset V^*$ is defined by the zeroes of the fundamental semi-invariant then we have $C \subset V^* \setminus V^*_{\mathrm{reg}}$ and the complement of $C \cup V^*_{\mathrm{reg}}$ in V^* has codimension ≥ 2. We write $d(V)$ for the degree of the fundamental semi-invariant. If $r(V) = 0$ (i.e. the action of $\mathfrak{g}$ on V is trivial) then we put $d(V) = 0$. We record the following

Lemma 2.1. *If $V = \mathfrak{g}$ then the fundamental semi-invariant in $S\mathfrak{g}$ is the square of the greatest common divisor of the Pfaffians of the principal $r(\mathfrak{g}) \times r(\mathfrak{g})$ minors in the structure matrix of $\mathfrak{g}$.*

Proof. According to [10] any $r(V)\times r(V)$-minor can be expressed as a quadratic form in Pfaffians of principal $r(V)\times r(V)$-minors. From this one easily deduces the stated result. □

In case $V = \mathfrak{g}$ is the adjoint representation then $r_{\mathfrak{g}}(\mathfrak{g}) = r(\mathfrak{g})$ is an even number and we we have $r(\mathfrak{g}) = \dim \mathfrak{g} - i(\mathfrak{g})$. We put

$$\begin{aligned} c(\mathfrak{g}) &= \frac{1}{2}(\dim \mathfrak{g} + i(\mathfrak{g})) \\ &= \dim \mathfrak{g} - \frac{1}{2} r(\mathfrak{g}) \end{aligned}$$

3. Reduction to the case without proper semi-invariants

The following result which generalizes [2, Thm. 1.19(3)] is the main result of this section.

Proposition 3.1. *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there exists another finite dimensional Lie algebra $\mathfrak{g}'$ such that $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g}')^{\mathfrak{g}'}_{\mathrm{si}} = (S\mathfrak{g}')^{\mathfrak{g}'}$. Moreover $c(\mathfrak{g}') = c(\mathfrak{g})$.*

If $\mathfrak{g}$ is almost algebraic then we may take $\mathfrak{g}'$ to be the intersection of the kernels of the non-trivial weights of the semi-invariants in $S\mathfrak{g}$ [1, 2, 7, 18]. This procedure must be modified for non-almost algebraic Lie algebras.

Example 3.2. Let $\mathfrak{g} = kv_1 + kv_2 + kv_3$ be the Lie algebra with non-trivial brackets $[v_1, v_2] = v_2 + v_3$, $[v_1, v_3] = v_3$. Then $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = k[v_3]$. On the other hand the kernel of the weight of v_3 is the abelian Lie algebra $kv_2 + kv_3$ whose semi-invariants are $k[v_2, v_3]$. So this is different. It turns out that in this case we have to take $\mathfrak{g}' = kv_1 + kv_2 + kv_3$ with non-trivial brackets $[v_1, v_2] = v_3$.

The proof of Proposition 3.1 will be given after some preparation.

Proposition 3.3. *Assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension one. Then one of the inclusions $(S\mathfrak{h})^{\mathfrak{g}}_{\mathrm{si}} \subset (S\mathfrak{h})^{\mathfrak{h}}_{\mathrm{si}}$ or $(S\mathfrak{h})^{\mathfrak{g}}_{\mathrm{si}} \subset (S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$ is an equality.*

This result is perhaps better appreciated in the following equivalent formulation.

Corollary 3.4. *Assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension one. Then either $(S\mathfrak{h})^{\mathfrak{h}}_{\mathrm{si}} \subset (S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$ or $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} \subset (S\mathfrak{h})^{\mathfrak{h}}_{\mathrm{si}}$*

Note that *both* inclusions may be equalities. This happens already for the two dimensional non-abelian Lie algebra.

Proof of Proposition 3.3. Assume that the statement is false. Write $\mathfrak{g}$ as a semi-direct product $\mathfrak{h}+kp$. Let f be a semi-invariant with weight χ in $(S\mathfrak{h})^{\mathfrak{h}}_{\mathrm{si}} \setminus (S\mathfrak{h})^{\mathfrak{g}}_{\mathrm{si}}$ which is irreducible in $S\mathfrak{h}$ and let $g = \sum_{i=0}^{n} a_i p^i$ be a semi-invariant with weight ψ in $S\mathfrak{g}$ such that $a_i \in S\mathfrak{h}$, $n > 0$ and $a_n \neq 0$.

Since f is not a semi-invariant for $\mathfrak{g}$ it is not a semi-invariant for p. From the fact that g is a semi-invariant for p we easily deduce that the a_i are semi-invariants for p. In particular the non-zero a_i cannot be divisible by f (since a factor of a semi-invariant for p is a semi-invariant for p).

We will now obtain a contradiction by computing the Poisson bracket $\{f, g\}$. Since g is a semi-invariant in $S\mathfrak{g}$ with weight ψ we have

$$\{f, g\} = \partial_\psi(f) \sum_i a_i p^i$$

Since f is a semi-invariant in $S\mathfrak{h}$ with weight χ we have

$$\{f, g\} = -\sum_i \partial_\chi(a_i) f p^i + \sum_i a_i i p^{i-1}\{f, p\}$$

Assume first $\partial_\psi(f) \neq 0$. Using the fact that $\partial_\psi(f)$ has lower degree than f and hence is not divisible by f we conclude that a_n is divisible by f. Since $a_n \neq 0$ this is a contradiction.

Assume now $\partial_\psi(f) = 0$. In that case we obtain from the fact that f does not divide $\{f, p\}$ (since f is not a semi-invariant for p) that f divides a_i for $i > 0$. This is again a contradiction. □

In the next two propositions we give some conditions under which the ring of semi-invariants for a representation does not change under passage to an ideal of the Lie algebra.

Proposition 3.5. *Let V be a finite dimensional representation of $\mathfrak{g}$. Assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ such that $r_{\mathfrak{h}}(V) = r_{\mathfrak{g}}(V)$. Then $(SV)^{\mathfrak{h}}_{\mathrm{si}} = (SV)^{\mathfrak{g}}_{\mathrm{si}}$.*

Proof. Assume $r_{\mathfrak{h}}(V) = r_{\mathfrak{g}}(V)$ and $(SV)^{\mathfrak{h}}_{\mathrm{si}} \neq (SV)^{\mathfrak{g}}_{\mathrm{si}}$. Let $f \in (SV)^{\mathfrak{h}}_{\mathrm{si}} \setminus (SV)^{\mathfrak{g}}_{\mathrm{si}}$ be a semi-invariant with weight χ which is irreducible in SV. Let $(h_i)_i$ be a basis of $\mathfrak{h}$ and let $p \in \mathfrak{g} - \mathfrak{h}$ be such that f is not a semi-invariant for p. Then by elementary linear algebra applied to the structure matrices of V with respect to $\mathfrak{g}$ and $\mathfrak{h}$ there exist a, $b_i \in SV$ with $a \neq 0$ such that $\delta = a \otimes p + \sum_i b_i \otimes h_i \in SV \otimes \mathfrak{g}$ has the property that $\rho(\delta)$ acts trivially on V. Hence $\delta \in \ker \rho$.

We claim we may choose δ in such a way that a is not divisible by f. Assume on the contrary that $a = f^n a'$, $n > 0$ such that f does not divide $a' \in SV$.

Since $\ker \rho$ is an ideal we have that $[1 \otimes p, \delta] \in \ker \rho$ and

$$[1 \otimes p, \delta] = p(a) \otimes p + \sum_i b'_i \otimes h_i$$

for suitable $b'_i \in SV$. Then $p(a) = n f^{n-1} p(f) a' + f^n p(a')$. Since $p(f)$ is not divisible by f (as it is not a semi-invariant for p) we see that the highest power of f which

divides $p(a)$ is f^{n-1} (and $p(a) \neq 0$). Replacing δ by $[1 \otimes p, \delta]$ and repeating this procedure we eventually arrive at a δ such that a is no longer divisible by f.

Applying this new δ to f we get

$$0 = ap(f) + \sum_i b_i h_i(f) = ap(f) + \sum b_i \chi(h_i) f$$

Since neither a nor $p(f)$ is divisible by f we have obtained a contradiction. □

Proposition 3.6. *Let V be a finite dimensional representation of $\mathfrak{g}$ and assume that $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ of codimension one such that $\mathfrak{g} = \mathfrak{h} + ks$ with s acting semi-simply on both V and $\mathfrak{h}$. Then $(SV)^{\mathfrak{h}}_{\mathrm{si}} = (SV)^{\mathfrak{g}}_{\mathrm{si}}$*

Proof. Put $S = SV$. We decompose $\mathfrak{h}$ and S according to the s-weights (i.e. as s-eigenspaces): $\mathfrak{h} = \oplus_{\mu \in k} \mathfrak{h}_\mu$, $S = \oplus_{\lambda \in k} S_\lambda$. For $f \in S$ let $\operatorname{Supp} f$ be the set of λ such that $f_\lambda \neq 0$ in the decomposition $f = \sum_{\lambda \in k} f_\lambda$ with $f_\lambda \in S_\lambda$.

Let $f \in S$ be a semi-invariant for $\mathfrak{h}$ with weight χ. Write $f = \sum_{\lambda \in k} f_\lambda$ $f_\lambda \in S_\lambda$. We claim that the f_λ are semi-invariants for $\mathfrak{h}$, which implies that they are in fact semi-invariants for $\mathfrak{g} = \mathfrak{h} + ks$. Hence $(SV)^{\mathfrak{h}}_{\mathrm{si}} \subset (SV)^{\mathfrak{g}}_{\mathrm{si}}$. Since the other inclusion is obvious we are done.

To prove the claim first assume $\chi = 0$. Thus $f \in S^{\mathfrak{h}}$. Pick $h \in \mathfrak{h}_\mu$. Then $0 = h(f) = \sum_{\lambda \in k} h(f_\lambda)$ with $h(f_\lambda) \in S_{\mu+\lambda}$. Hence $h(f_\lambda) = 0$ and thus $f_\lambda \in S^{\mathfrak{h}}$. So this case is OK.

Now assume $\chi \neq 0$. We first assert that $\chi(\mathfrak{h}_\mu) = 0$ for $\mu \neq 0$. To see this assume there exist $h \in \mathfrak{h}_\mu$, $\mu \neq 0$ such that $\chi(h) \neq 0$. From the equation $h(f) = \chi(h) f$ we deduce $\operatorname{Supp} f = \operatorname{Supp}(h(f)) \subset \mu + \operatorname{Supp} f$ which is impossible if $\mu \neq 0$. So our assertion is correct.

As in the case $\chi = 0$ we now deduce for $h \in \mathfrak{h}_\mu$ that $h(f_\lambda) = 0$ if $\mu \neq 0$ and $h(f_\lambda) = \chi(h) f_\lambda$, if $\mu = 0$. So the f_λ's are semi-invariants in this case also. □

Lemma 3.7. *Assume that $f \in S\mathfrak{g}$ is a semi-invariant with weight χ and $\mathfrak{h} = \ker \chi$. Then $c(\mathfrak{g}) = c(\mathfrak{h})$.*

Proof. We may assume $\mathfrak{g} \neq \mathfrak{h}$, i.e. χ is non-trivial. Assume $c(\mathfrak{g}) \neq c(\mathfrak{h})$. Choose $p \in \mathfrak{g}$ such that $\chi(p) = 1$. Comparing

$$c(\mathfrak{g}) = \dim \mathfrak{g} - \frac{1}{2} r(\mathfrak{g})$$
$$c(\mathfrak{h}) = \dim \mathfrak{g} - \frac{1}{2} r(\mathfrak{h}) - 1$$

we see that

$$r(\mathfrak{g}) \neq r(\mathfrak{h}) + 2$$

Since the structure matrix of $\mathfrak{g}$ is obtained from that of $\mathfrak{h}$ by adding a row and a column we have $r(\mathfrak{g}) - r(\mathfrak{h}) \in \{0, 2\}$. We obtain

$$r(\mathfrak{g}) = r(\mathfrak{h})$$

The proof now parallels that of Lemma 3.5. Let $(h_i)_i$ be a basis of $\mathfrak{h}$ and select a, $b_i \in S\mathfrak{g}$ with $a \neq 0$ such that $\delta = a \otimes p + \sum_i b_i \otimes h_i$ acts trivially on p and $(h_j)_j$. Thus $\delta \in \ker \rho$. In other words $\rho(\delta)$ acts trivially on $S\mathfrak{g}$. But we also find $\rho(\delta)(f) = af \neq 0$. This is a contradiction. □

Proof of Proposition 3.1. We will construct $\mathfrak{g}'$ one step at a time. Assume that $\mathfrak{g}$ has a non-trivial weight χ on $S\mathfrak{g}$. Then we will construct a Lie algebra $\tilde{\mathfrak{g}}$ such that $c(\tilde{\mathfrak{g}}) = c(\mathfrak{g})$, $(S\mathfrak{g})^{\mathfrak{g}}_{\text{si}} = (S\tilde{\mathfrak{g}})^{\tilde{\mathfrak{g}}}_{\text{si}}$ and such that either $\dim\tilde{\mathfrak{g}} < \dim\mathfrak{g}$ or $\dim N(\tilde{\mathfrak{g}}) > \dim N(\mathfrak{g})$ where $N(-)$ denotes the nil-radical. It is clear that by repeating this procedure we eventually end up with a Lie algebra which has the requested properties.

Let $f \in S\mathfrak{g}$ be an non-zero eigenfunction for χ. Put $\mathfrak{h} = \ker\chi$. Since semi-invariants Poisson commute and since the Poisson centralizer of f is equal to $S\mathfrak{h}$ we find (see [2, Cor. 1.15]).

$$(S\mathfrak{g})^{\mathfrak{g}}_{\text{si}} \subset (S\mathfrak{h})^{\mathfrak{h}}_{\text{si}} \tag{3.1}$$

Choose $c \in \mathfrak{g}$ such that $\chi(c) = 1$. Then $\text{ad}(c) = D_s + D_p$ where D_s, D_p are two commuting derivations of $\mathfrak{h}$ with D_s being semi-simple and D_p nilpotent. Let $\mathfrak{j} = \mathfrak{h}+ks+kp$ be the semi-direct product of $\mathfrak{h}$ with an abelian Lie algebra $ks+kp$ such that $\text{ad}(s)$ acts by D_s and $\text{ad}(p)$ acts by D_p. Sending c to $s+p$ yields an embedding $\mathfrak{g} \subset \mathfrak{j}$. Put $\mathfrak{k} = \mathfrak{h} + kp$. Then we have $\mathfrak{j} = \mathfrak{g} + ks = \mathfrak{k} + ks$.

Since $\text{ad}(s)$ acts semi-simply on everything we have by Proposition 3.6

$$(S\mathfrak{g})^{\mathfrak{g}}_{\text{si}} = (S\mathfrak{g})^{\mathfrak{j}}_{\text{si}} = (S\mathfrak{g})^{\mathfrak{k}}_{\text{si}}$$

and thus by (3.1)

$$(S\mathfrak{g})^{\mathfrak{g}}_{\text{si}} = S\mathfrak{h} \cap (S\mathfrak{g})^{\mathfrak{k}}_{\text{si}} = (S\mathfrak{h})^{\mathfrak{k}}_{\text{si}}$$

By Proposition 3.3 we have either $(S\mathfrak{h})^{\mathfrak{k}}_{\text{si}} = (S\mathfrak{h})^{\mathfrak{h}}_{\text{si}}$ or $(S\mathfrak{h})^{\mathfrak{k}}_{\text{si}} = (S\mathfrak{k})^{\mathfrak{k}}_{\text{si}}$.

Assume first $(S\mathfrak{h})^{\mathfrak{k}}_{\text{si}} = (S\mathfrak{h})^{\mathfrak{h}}_{\text{si}}$. Then we put $\tilde{\mathfrak{g}} = \mathfrak{h}$. By Lemma 3.7 we have $c(\tilde{\mathfrak{g}}) = c(\mathfrak{g})$. Since $\dim\tilde{\mathfrak{g}} < \dim\mathfrak{g}$ this case is done.

Now assume $(S\mathfrak{h})^{\mathfrak{k}}_{\text{si}} \neq (S\mathfrak{h})^{\mathfrak{h}}_{\text{si}}$ and thus $(S\mathfrak{h})^{\mathfrak{k}}_{\text{si}} = (S\mathfrak{k})^{\mathfrak{k}}_{\text{si}}$. In this case we put $\tilde{\mathfrak{g}} = \mathfrak{k}$ and hence we have $\dim\tilde{\mathfrak{g}} = \dim\mathfrak{g}$. By Proposition 3.5 we have $r_{\mathfrak{k}}(\mathfrak{h}) > r_{\mathfrak{h}}(\mathfrak{h})$ and hence $r(\mathfrak{k}) = r_{\mathfrak{k}}(\mathfrak{k}) > r_{\mathfrak{h}}(\mathfrak{h}) = r(\mathfrak{h})$.

If $r(\mathfrak{g}) = r(\mathfrak{h})$ then $r_{\mathfrak{h}}(\mathfrak{g}) = r_{\mathfrak{h}}(\mathfrak{h})$ and hence by Proposition 3.5 $(S\mathfrak{h})^{\mathfrak{g}}_{\text{si}} = (S\mathfrak{h})^{\mathfrak{h}}_{\text{si}}$. Since $(S\mathfrak{h})^{\mathfrak{g}}_{\text{si}} = (S\mathfrak{g})^{\mathfrak{g}}_{\text{si}} = (S\mathfrak{h})^{\mathfrak{k}}_{\text{si}}$ this is a contradiction. Thus $r(\mathfrak{g}) > r(\mathfrak{h})$. Since the ranks involved jump at most by 2 we deduce $r(\mathfrak{g}) = r(\mathfrak{k})$ and hence $c(\tilde{\mathfrak{g}}) = c(\mathfrak{g})$.

It remains to show that in this case we have $\dim N(\mathfrak{g}) < \dim N(\tilde{\mathfrak{g}})$. Since p acts nilpotently we have $N(\tilde{\mathfrak{g}}) = kp+N(\mathfrak{h})$. We claim that $N(\mathfrak{g}) \subset N(\mathfrak{h})$ which is sufficient. To prove this claim we need to show that no element of the form $c+n$ for $n \in \mathfrak{h}$ acts nilpotently on $\mathfrak{h}$. Assume such n exists. Then $0 = \chi(c+n) = \chi(c)$ and hence $c \in \mathfrak{h}$ which is a contradiction. $\square$

4. A formula for the transcendence degree of invariants

If S is a commutative domain then we denote its field of fractions by $Q(S)$. In this section we prove the following result.

Proposition 4.1. *Let V be a finite dimensional representation of $\mathfrak{g}$ and assume that SV contains no proper semi-invariants. Then*

$$\operatorname{trdeg} Q(SV)^{\mathfrak{g}} = \dim V - r(V) \tag{4.1}$$

In the case that $\mathfrak{g}$ acts algebraically on V the formula (4.1) was proved by Dixmier [5, Lemme 7] (it is a more or less direct consequence of Rosenlicht's theorem [20]). Here we have traded algebraicity for the absence of proper semi-invariants. Both conditions are independent as Example 4.8 below shows.

Let L/k be a finitely generated field extension on which $\mathfrak{g}$ acts by derivations. Put $K = L^{\mathfrak{g}}$. It is clear that K is algebraically closed in L. We will say that the action is *geometric* if the induced map $\rho : L \otimes_K \mathfrak{g} \to \operatorname{Der}_K(L)$ is surjective. If $x_1, \ldots, x_n$ is a transcendence basis of L/K and $\partial_i = \partial/\partial x_i : L \to L$ are the corresponding derivations then $\operatorname{Der}_K(L) = \sum_i L\partial_i$. From this we deduce

Lemma 4.2. *Let L be as above. Then the action is geometric if one has*

$$\operatorname{trdeg}_{L^{\mathfrak{g}}} L = \dim \mathfrak{g} - \dim_L \ker \rho \tag{4.2}$$

Note that $\ker \rho$ can also be computed as $\ker(L \otimes_k \mathfrak{g} \to \operatorname{Der}_k(L))$. So the number on the right hand side of (4.2) can be computed without knowing $L^{\mathfrak{g}}$.

Lemma 4.3. *Let $M \supset L \supset K$ be finitely generated field extensions of k and let $\mathfrak{g}$ be a finite dimensional Lie algebra acting on M such that $K = M^{\mathfrak{g}}$. Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$ and put $L = M^{\mathfrak{h}}$. Assume that the $\mathfrak{h}$-action on M is geometric and likewise for the action $\mathfrak{g}/\mathfrak{h}$ on L. Then the action of $\mathfrak{g}$ on M is geometric.*

Proof. This follows from the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M \otimes_k \mathfrak{h} & \longrightarrow & M \otimes_k \mathfrak{g} & \longrightarrow & M \otimes_k \mathfrak{g}/\mathfrak{h} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Der}_L(M) & \longrightarrow & \operatorname{Der}_K(M) & \longrightarrow & M \otimes_L \operatorname{Der}_K(L) & \longrightarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

□

Now let S be a k-algebra which is an integral domain with finitely generated fraction field. Assume $\mathfrak{g}$ acts on S. We say that $\mathfrak{g}$ acts *generically geometrically* if the induced action on the fraction field is geometric.

If Y is a (possibly singular) variety then we denote the tangent space in a point $y \in Y$ by $T_{Y,y}$. If Y is smooth then $T_{Y,y}$ is the fiber of the tangent bundle T_Y of Y at y.

Proposition 4.4. *Assume that S/k is a finitely generated domain and that $\mathfrak{g}$ acts on S. Put $Y = \operatorname{Spec} S$. Let L be the fraction field of S. Then the action is generically geometric if and only if*

$$\operatorname{trdeg}_{L^{\mathfrak{g}}} L = \dim \mathfrak{g} - \min_{y \in Y} \dim \mathfrak{g}_y$$

where the minimum is taken over the closed points in Y and $\mathfrak{g}_y$ denotes the stabilizer of $y \in Y$, i.e. the kernel of $\mathfrak{g} \to T_{Y,y}$.

Proof. For technical reasons it is more convenient to work with differentials instead of with vector fields as the sheaf of differentials is always compatible with taking fibers.

There is a canonical pairing $\mathfrak{g} \otimes_k \Omega_Y \to \mathcal{O}_Y : v \otimes f\,dg \mapsto fv(g)$ which yields a map of coherent $\mathcal{O}_Y$ modules $\rho^* : \Omega_Y \to \mathcal{O}_Y \otimes \mathfrak{g}^*$. Taking the fiber in a point $y \in Y$ one checks $\mathfrak{g}_y^* = \operatorname{coker} \rho_y^*$. By semi-continuity the dimension of the cokernel of the generic fiber of ρ^* is equal to the minimum of the dimensions of the cokernels of the special fibers.

Thus we find

$$\begin{aligned}\dim_L \ker(L\otimes_k \mathfrak{g}\to \mathrm{Der}_k(L)) &= \dim_L \operatorname{coker}(\Omega_{L,k}\to L\otimes_k \mathfrak{g}^*)\\ &= \min_{y\in Y} \dim \mathfrak{g}_y^*\\ &= \min_{y\in Y} \dim \mathfrak{g}_y\end{aligned}$$

It now suffices to apply Lemma 4.2. □

Lemma 4.5. *Assume that $\mathfrak{g} = Lie(G)$ where G is a connected algebraic group acting rationally on a domain S with finitely generated fraction field. Then the action is generically geometric.*

Proof. Let L be the fraction field of S. Since L is finitely generated and since G acts rationally on S we may select a finitely generated G-invariant subring $S_0\subset S$ such that L is the fraction field of S_0. From here on the proof proceeds as in [3, Lemme 7]. For the benefit of the reader let us repeat the argument. By Rosenlicht's theorem there exists an G-invariant open $U\subset Y=\operatorname{Spec} S_0$ such that a geometric quotient U/G exists. Shrinking U we may assume that U is smooth. By the properties of a geometric quotient the field of rational functions on U is L, the field of rational functions on U/G is L^G and the fibers of $U\to U/G$ are the G-orbits. Hence $\operatorname{trdeg} L/L^G = \dim U-\dim U/G$ is equal to the dimension of the generic G orbit on U or equivalently on Y. This is $\dim G-\min_{y\in Y}\dim G_y = \dim \mathfrak{g}-\min_{y\in Y}\dim \mathfrak{g}_y$. We may now apply Proposition 4.4 with $S=S_0$. □

We would like to have a transitivity result as in Lemma 4.3 but one does not always have $Q(S)^{\mathfrak{g}} = Q(S^{\mathfrak{g}})$. So we introduce a special situation in which this identity holds.

Let us say that a graded ring S is *connected* if it is of the form $k+S_1+S_2+\cdots$ with $\dim S_i<\infty$. For technical reasons we do not assume that S is finitely generated. Recall the following.

Lemma 4.6. *Let S be a connected graded factorial domain, and assume that $\mathfrak{g}$ acts in a graded way on S without proper semi-invariants. Then $S^{\mathfrak{g}}$ is factorial, and furthermore $Q(S)^{\mathfrak{g}} = Q(S^{\mathfrak{g}})$.*

The following is a weak version of Rosenlicht's theorem which is also valid for non-algebraic Lie algebras.

Proposition 4.7. *Let S be a connected factorial graded domain with finitely generated quotient field and assume that $\mathfrak{g}$ acts in a graded way on S without proper semi-invariants. Then the action is generically geometric.*

Proof. We have a filtration by ideals

$$0=\mathfrak{g}_0\subset\cdots\subset\mathfrak{g}_{n-1}\subset\mathfrak{g}_n=\mathfrak{g}$$

where $\mathfrak{g}_n/\mathfrak{g}_{n-1}$ is semi-simple, and the other quotients are abelian. We have a corresponding filtration

$$S=S^0\supset\cdots\supset S^{n-1}\supset S^n=S^{\mathfrak{g}}$$

with $S^i=S^{\mathfrak{g}_i}$ and hence $S^{i+1}=(S^i)^{\mathfrak{g}_{i+1}/\mathfrak{g}_i}$.

By Lemmas 4.6 and 4.3 we may assume that $\mathfrak{g}$ is either semi-simple or abelian. If $\mathfrak{g}$ is semi-simple then it acts algebraically (since $\dim S_i<\infty$) and hence we may invoke Lemma 4.5. If $\mathfrak{g}$ is abelian then its generalized weights must be zero

(for otherwise we could construct a proper semi-invariant). Hence $\mathfrak{g}$ acts locally nilpotently and hence algebraically. We may again invoke Lemma 4.5. $\square$

Proof of Proposition 4.1. Put $L = Q(SV)$. By (4.2) $\operatorname{trdeg}_{L^\mathfrak{g}} L = \dim \mathfrak{g} - n$ where n is the dimension of the null space of the structure matrix. The structure matrix has size $\dim V \times \dim \mathfrak{g}$. Thus $\operatorname{trdeg}_{L^\mathfrak{g}} L = r(V)$. Hence $\operatorname{trdeg} L^\mathfrak{g} = \dim V - r(V)$. $\square$

Example 4.8. Let W be a finite dimensional vector space. Then the *Heisenberg Lie algebra* $\mathfrak{h}$ on W is the vector space $W \oplus W^* \oplus kc$ with for all $w, w' \in W$, $\phi, \phi' \in W^*$: $[w, w'] = 0$, $[\phi, \phi'] = 0$, $[c, w] = 0$, $[c, \phi] = 0$, $[\phi, w] = \phi(w)c$.

For $p \in \operatorname{End}(W)$ let D_p be the derivation $(p, -p^*, 0)$ of $\mathfrak{h}$ and let $\mathfrak{g} = \mathfrak{h} + kt$ be the corresponding semi-direct product. Put $S = S\mathfrak{g}$. Assume that p is invertible. Then $\mathfrak{g}$ has a non-degenerate invariant symmetric bilinear form $(-,-)$ whose non-trivial values are given by $(t, c) = 1$, $(\phi, w) = -\phi(p^{-1}(w))$. Hence $\mathfrak{g}$ is quadratic. It then follows from [15, Cor. 2.3, Prop. 3.2] that $S\mathfrak{g}$ contains no proper semi-invariants. However if we choose p to be non-diagonalizable then $\mathfrak{g}$ does not act algebraically as t does not have a Jordan decomposition. So the hypotheses of Proposition 4.1 do not imply that $\mathfrak{g}$ acts algebraically.

The formula (4.1) asserts that the transcendence degree of $Q(S)^\mathfrak{g}$ should be 2, as an easy computation shows $r(\mathfrak{g}) = \dim \mathfrak{g} - 2$.

This example is simple enough to verify directly. As an aside we find that the hypothesis that p is invertible is in fact superfluous. The absence of proper semi-invariants always holds.

Choose a basis $(w_i)_i$ for W and assume that $p(w_i) = \sum_j p_{ij} w_j$. Then one has $p^*(w_j^*) = \sum_i p_{ij} w_i^*$. Put $z = tc - \sum_{ij} p_{ij} w_i w_j^*$ (this is the Casimir element for the pairing $(-,-)$ in the case that p is invertible). Then $c, z \in S^\mathfrak{g}$. We write $S_c = k[c^{\pm 1}, z, (w_i c^{-1})_i, w_j^*]$. With respect to these new generators the only non-trivial Poisson brackets are $\{w_j^*, w_i c^{-1}\} = \delta_{ji}$. A semi-invariant in S generates a principal Poisson ideal in S and hence in S_c. Computing with the new generators we see that the only principal Poisson ideals in S_c are those generated by elements in $k[c^{\pm 1}, z]$. Thus $S^\mathfrak{g}_{\mathrm{si}} = S \cap k[c^{\pm 1}, z]$ which is equal to $k[c, z]$ if $p \neq 0$ and equal to $k[c, t]$ otherwise. Thus we see that $S\mathfrak{g}$ contains no proper semi-invariants. In both cases we find $\operatorname{trdeg} Q(S)^\mathfrak{g} = 2$ as predicted by (4.1).

5. Proofs in the absence of proper semi-invariants

Throughout V is a finite dimensional representation of $\mathfrak{g}$. For convenience we write $S = SV$, $R = (SV)^\mathfrak{g}$ and we let L be the field of fractions of S. As we will mostly use geometrical language we also put $Y = \operatorname{Spec} S = V^*$, $X = \operatorname{Spec} R$ and we let $\pi : Y \to X$ be dual to the inclusion $R \to S$. If R is finitely generated then the regular locus of X is denoted[1] by X_{sm}.

The following result is an adaptation of [12] to the case of non-semisimple Lie algebras.

Proposition 5.1. *Assume that $(SV)^\mathfrak{g}_{\mathrm{si}} = (SV)^\mathfrak{g}$ and that $(SV)^\mathfrak{g}$ is finitely generated. Let*

$$U = \{y \in Y \mid \pi(y) \in X_{\mathrm{sm}} \textit{ and } \pi \textit{ is smooth in } y\}$$

Then $\operatorname{codim}_Y(Y - U) \geq 2$.

[1]Unfortunately the subscript "reg" is already taken by $\mathfrak{g}^*_{\mathrm{reg}}$.

Proof. The proof is that of [12] with minor adaptations. Without loss of generality we may replace $\mathfrak{g}$ by the algebraic hull of the image of $\mathfrak{g}$ in $\mathrm{End}_k(V)$. Let $G \subset \mathrm{GL}(V)$ be the affine connected algebraic group such that $\mathrm{Lie}\, G = \mathfrak{g}$. Then G acts rationally on S and $R = S^G$.

Let E be the union of the irreducible divisors in $Y - U$. Then E is G-invariant. Since G is connected it follows that E is irreducible. Since S is factorial it follows that $E = V(f)$ for some irreducible $f \in S$.

For $\sigma \in G$ we have that $c_\sigma = \sigma(f)f^{-1}$ is a unit in S and hence $c_\sigma \in k^*$. Thus f is a semi-invariant and hence $f \in R$.

We claim that the map $R/fR \to S/fS$ is injective. Assume there is some element $\bar{c}$ in the kernel. Then $c = fd$ with $d \in S$. But then $d \in S^{\mathfrak{g}} = R$. Hence $\bar{c} = 0$.

Let D be the divisor in X of f. Then $E = \pi^{-1}(D)$ and the map $E \to D$ is a dominant map between irreducible algebraic varieties. Since X is normal $D \cap X_{\mathrm{sm}} \neq \emptyset$.

By generic smoothness there exist dense open $E' \subset E$, $D' \subset D \cap X_{\mathrm{sm}}$ such that E', D' are regular and π restricts to a smooth map $E' \to D'$.

Let $y \in E'$. We will show that π is smooth at y, contradicting the fact that E' is contained in the non smooth locus of π. We consider the following commutative diagram of tangent spaces with $x = \pi(y)$

$$\begin{tikzcd}
0 \arrow[r] & T_{E',y} \arrow[r] \arrow[d, "d\pi|_{E'}"'] & T_{Y,y} \arrow[r, "df_y"] \arrow[d, "d\pi"'] & k \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & T_{D',x} \arrow[r] & T_{X,x} \arrow[r, "df_x"'] & k \arrow[r] & 0
\end{tikzcd}$$

Since x is regular in D, X and y is regular in E, Y the rows are exact. The left most map is surjective since $E' \to D'$ is smooth. This implies that the middle map is surjective. $\square$

We keep the notations as in the statement of Proposition 5.1 and we assume throughout that $(SV)^{\mathfrak{g}}_{\mathrm{si}} = (SV)^{\mathfrak{g}}$ and that $(SV)^{\mathfrak{g}}$ is finitely generated. This implies in particular that $(SV)^{\mathfrak{g}}$ is factorial and $L^{\mathfrak{g}} = Q((SV)^{\mathfrak{g}})$. Furthermore by Propositions 4.7 and 4.4 we have

$$\dim Y - \dim X = \operatorname{trdeg}_{L^{\mathfrak{g}}} L = \dim \mathfrak{g} - \min_{y \in Y} \dim \mathfrak{g}_y \tag{5.1}$$

This leads to the definition

$$Y' = \{ y \in Y \mid \dim \mathfrak{g} - \dim \mathfrak{g}_y < \dim Y - \dim X \} \subsetneq Y$$

and we will also put

$$W = U \cap (Y - Y') \tag{5.2}$$

Since the elements of $\mathfrak{g}$ define vector fields on Y which annihilate invariant functions, we have the usual map of vector bundles on Y

$$\rho : \mathcal{O}_Y \otimes_k \mathfrak{g} \to T_Y$$

which extends to a complex of vector bundles on $\pi^{-1} X_{\mathrm{sm}}$

$$\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes_k \mathfrak{g} \xrightarrow{\rho_{\pi^{-1}X_{\mathrm{sm}}}} T_{\pi^{-1}X_{\mathrm{sm}}} \xrightarrow{d\pi_{\pi^{-1}X_{\mathrm{sm}}}} \pi^*_{\pi^{-1}X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}} \to 0 \tag{5.3}$$

We see that U is the locus in $\pi^{-1}X_{\mathrm{sm}}$ where (5.3) is exact at $\pi^*_{\pi^{-1}X_{\mathrm{sm}}}T_{X_{\mathrm{sm}}}$ and W is the locus where the entire complex is exact. Thus we have an exact sequence

$$\mathcal{O}_W \otimes_k \mathfrak{g} \xrightarrow{\rho_W} T_W \xrightarrow{d\pi_W} \pi^*_W T_{X_{\mathrm{sm}}} \to 0 \tag{5.4}$$

The following is one of the main results of [12]. For completeness we include the proof in our setting.

Proposition 5.2. *One has $W = (Y - Y') \cap \pi^{-1}X_{\mathrm{sm}}$.*

The statement of this Proposition means that if $y \in Y$ is such that $\pi(y) = x$ is regular in X and $\mathfrak{g}_y$ has minimal dimension then π is smooth in y.

Proof of Proposition 5.2. Put $\overline{W} = (Y - Y') \cap \pi^{-1}X_{\mathrm{sm}}$. Since $W = U \cap (Y - Y')$ the inclusion $W \subset \overline{W}$ is obvious. To prove the opposite inclusion we look at the complex

$$\mathcal{O}_{\overline{W}} \otimes_k \mathfrak{g} \xrightarrow{\rho_{\overline{W}}} T_{\overline{W}} \xrightarrow{d\pi_{\overline{W}}} \pi^*_{\overline{W}} T_{X_{\mathrm{sm}}} \to 0$$

Since $\rho_{\overline{W}}$ has constant rank $\operatorname{coker}\rho_{\overline{W}}$ is a vector bundle. Furthermore by the exactness of (5.4) we deduce that $\operatorname{coker}\rho_{\overline{W}} \to \pi^*_{\overline{W}} T_{X_{\mathrm{sm}}}$ is an isomorphism on W.

Since $W \subset \overline{W} \subset Y - Y'$ and $Y - Y' - W = (Y - Y') \cap (Y - U)$ has codimension ≥ 2 in $Y - Y'$, the same holds for the codimension of $\overline{W} - W$ in $\overline{W}$. It follows that $\operatorname{coker}\rho_{\overline{W}} \to \pi^*_{\overline{W}} T_{X_{\mathrm{sm}}}$ is an isomorphism on the whole of $\overline{W}$. In particular the map $T_{\overline{W}} \to \pi^*_{\overline{W}} T_{X_{\mathrm{sm}}}$ is surjective. This implies that $\overline{W} \subset U$. Hence $\overline{W} = W$. □

Lemma 5.3. *(see also [13, Lemma 4]) The dual of (5.3) yields an exact sequence*

$$0 \to \pi^*_{\pi^{-1}X_{\mathrm{sm}}}\Omega_{X_{\mathrm{sm}}} \xrightarrow{d\pi^*_{\pi^{-1}X_{\mathrm{sm}}}} \Omega_{\pi^{-1}X_{\mathrm{sm}}} \xrightarrow{\rho^*_{\pi^{-1}X_{\mathrm{sm}}}} \mathfrak{g}^* \otimes \mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \tag{5.5}$$

of vector bundles on $\pi^{-1}X_{\mathrm{sm}}$.

Proof. Let $C = \operatorname{coker}\rho_{\pi^{-1}X_{\mathrm{sm}}}$. Then we have commutative diagram

$$\begin{tikzcd}
\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes_k \mathfrak{g} \arrow[r, "\rho_{\pi^{-1}X_{\mathrm{sm}}}"] \arrow[d, equal] & T_{\pi^{-1}X_{\mathrm{sm}}} \arrow[r, "d\pi_{\pi^{-1}X_{\mathrm{sm}}}"] \arrow[d, equal] & \pi^*_{\pi^{-1}X_{\mathrm{sm}}}T_{X_{\mathrm{sm}}} & \\
\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes_k \mathfrak{g} \arrow[r, "\rho_{\pi^{-1}X_{\mathrm{sm}}}"'] & T_{\pi^{-1}X_{\mathrm{sm}}} \arrow[r] & C \arrow[u, "\beta"'] \arrow[r] & 0 \\
 & & K \arrow[u] & \\
 & & 0 \arrow[u] &
\end{tikzcd}$$

Since $d\pi_{\pi^{-1}X_{\mathrm{sm}}}$ is surjective on U, the same holds for β. So the vertical sequence is exact on U. Furthermore since the upper sequence is exact on W it follows that K is torsion. Dualizing the vertical sequence we obtain $C^*|U \cong (\pi^*_{\pi^{-1}X_{\mathrm{sm}}}T_{X_{\mathrm{sm}}})^*|U$. This extends to an isomorphism $C^* = (\pi^*_{\pi^{-1}X_{\mathrm{sm}}}T_{X_{\mathrm{sm}}})^*$.

The lemma now follows by dualizing the lower exact sequence. □

We can now prove a generalization of Proposition 1.6(1) (taking into account that if $V = \mathfrak{g}$ we have $\rho^* = -\rho$, see below).

Proposition 5.4. *Assume that $(SV)^{\mathfrak{g}}_{\mathrm{si}} = (SV)^{\mathfrak{g}}$, and that $(SV)^{\mathfrak{g}}$ is a (necessarily finitely generated) polynomial ring. Then $\ker \rho^*$ is free.*

Proof. This follows immediately from Lemma 5.3, taking into account that $X = X_{\mathrm{sm}}$. □

Now we specialize to the case $V = \mathfrak{g}$ but we still assume that the conditions of Proposition 5.1 hold. Our assumption that $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$ implies in particular that $\mathfrak{g}$ is unimodular by [6]. To lighten the notations we silently fix an isomorphism $\wedge^{\dim \mathfrak{g}}\mathfrak{g} \cong k$.

In what follows we need to keep track of the grading. Therefore we introduce an additional 1-dimensional torus $\mathbb{G}_m$ which acts with weight n on the degree n-part of S. Everything we do is equivariant with respect to this torus. If L is the one dimensional representation of $\mathbb{G}_m$ corresponding to the identity character then we write $?(n)$ for $? \otimes L^n$.

Taking into account $V = \mathfrak{g}$ we obtain $T_Y = (\mathcal{O}_Y \otimes \mathfrak{g}^*)(1)$, $\Omega_Y = (\mathcal{O}_Y \otimes \mathfrak{g})(-1)$. In particular ρ may be viewed as a map:

$$\rho : \mathcal{O}_Y \otimes \mathfrak{g} \to (\mathcal{O}_Y \otimes \mathfrak{g}^*)(1)$$

Since ρ is represented by the structure matrix of $\mathfrak{g}$ which is anti-symmetric we obtain $\rho^* = -\rho(-1)$. Concatenating (5.3) with (5.5) we obtain a complex

(5.6)

$$0 \to (\pi^*_{\pi^{-1}X_{\mathrm{sm}}}\Omega_{X_{\mathrm{sm}}})(1) \xrightarrow{d\pi^*} \mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes_k \mathfrak{g} \xrightarrow{\rho} (\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes \mathfrak{g}^*)(1) \xrightarrow{d\pi} \pi^*_{\pi^{-1}X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}} \to 0$$

which is possibly non-exact at $(\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes \mathfrak{g}^*)(1)$ and $\pi^*_{\pi^{-1}X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}}$. The locus where the right non-trivial map is surjective is U. The locus where it is exact is precisely W.

If Z is a normal variety and F is a coherent torsion free $\mathcal{O}_Z$-module then we put $\det F = (\wedge^{\mathrm{rk}\, F} F)^{**}$. The operation $F \mapsto \det F$ is multiplicative on complexes which are exact in codimension ≥ 2. Furthermore the following is well known.

Lemma 5.5. *Let $\rho : F \to G$ be a map between vector bundles on Z which is generically of rank $r > 0$. Let $M = \operatorname{im} \rho$ and let $N \subset G$ be the maximal coherent subsheaf of G containing M such that N/M is torsion. Finally let $I(\rho)$ be the ideal in $\mathcal{O}_Z$ locally generated by the $r \times r$ minors in a matrix representation of ρ. Then $\det M = (I(\rho) \det N)^{**}$ as submodules of $\wedge^r G$.*

Proof. We may reduce to the case that Z is the spectrum of a discrete valuation ring D, $F = D^p$, $G = D^q$. In that case ρ may be diagonalized as

$$\begin{pmatrix}
\pi^{a_1} & 0 & \cdots & 0 & 0 & \cdots & 0\\
0 & \pi^{a_2} & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & \pi^{a_r} & 0 & \cdots & 0\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots & \vdots & & \vdots\\
0 & 0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix}$$

where π is a uniformizing element of D. Thus $I(\rho) = (\pi^{\sum_i a_i})$. We find $M = \pi^{a_1}D \oplus \cdots \oplus \pi^{a_r}D \oplus 0 \oplus \cdots \oplus 0 \subset D^q$ and $N = D^r \oplus 0 \oplus \cdots \oplus 0 \subset D^q$. Let $(e_i)_i$ be the standard basis of D^q. Then $\det M = \pi^{\sum_i a_i} De_1 \wedge \cdots \wedge e_r$, $\det N = De_1 \wedge \cdots \wedge e_r$ and so we have indeed $\det M = I(\rho) \det N$ inside $\wedge^r G$. □

We put $\omega_Z = \det \Omega_Z$. This is the so-called *dualizing module* on Z.

Put $M = \operatorname{im} \rho$, $N = \ker d\pi$ in (5.6). We obtain two exact sequences of torsion free $\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}}$-modules.

$$0 \to (\pi^*_{\pi^{-1}X_{\mathrm{sm}}} \Omega_{X_{\mathrm{sm}}})(1) \xrightarrow{d\pi^*} \mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes_k \mathfrak{g} \to M \to 0$$

$$0 \to N \to (\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes \mathfrak{g}^*)(1) \xrightarrow{d\pi} \pi^*_{\pi^{-1}X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}}$$

where $\operatorname{coker} d\pi$ is supported on $\pi^{-1}X_{\mathrm{sm}} - U$ which has codimension ≥ 2 by Proposition 5.1. We have $M \subset N$ and furthermore the support of N/M is $\pi^{-1}X_{\mathrm{sm}} - W$. Since $\pi^*_{\pi^{-1}X_{\mathrm{sm}}} T_{X_{\mathrm{sm}}}$ is torsion free we find that N is the maximal submodule of $(\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}} \otimes \mathfrak{g}^*)(1)$ containing M such that N/M is torsion. Hence we are in the setting of Lemma 5.5, provided $\mathfrak{g}$ is non-abelian (we want $\operatorname{rk} \rho > 0$), which we will temporarily assume. Let f be the fundamental semi-invariant (cfr §2) in $S\mathfrak{g}$. Then we have $I(\rho) \subset (f)$ and $(f)/I(\rho)$ is supported in codimension ≥ 2. Hence in the application of Lemma 5.5 we may replace $I(\rho)$ by $(f) = \mathcal{O}_Y(-d(\mathfrak{g}))$.

Taking into account $\dim \mathfrak{g} = \dim Y$ we conclude (using multiplicativity of det)

$$\det M = (\pi^*_{\pi^{-1}X_{\mathrm{sm}}} \omega_{X_{\mathrm{sm}}})^*(-\dim X)$$
$$\det N = (\pi^*_{\pi^{-1}X_{\mathrm{sm}}} \omega_{X_{\mathrm{sm}}})(\dim Y)$$

Hence by Lemma 5.5 we obtain

$$(\pi^*_{\pi^{-1}X_{\mathrm{sm}}} \omega_{X_{\mathrm{sm}}})^*(-\dim X) = (\pi^*_{\pi^{-1}X_{\mathrm{sm}}} \omega_{X_{\mathrm{sm}}})(\dim Y - d(\mathfrak{g}))$$

or in other words

$$\mathcal{O}_{\pi^{-1}X_{\mathrm{sm}}}(-\dim Y - \dim X + d(\mathfrak{g})) = \pi^*_{\pi^{-1}X_{\mathrm{sm}}} \omega^{\otimes 2}_{X_{\mathrm{sm}}} \tag{5.7}$$

If $T = k + T_1 + \cdots +$ is a finitely generated positively graded normal commutative ring and if $\omega_T \cong T(-a)$ then we will call a the *Gorenstein invariant* of T and denote it by $a(T)$. This is for example always defined if T is factorial.[2]

Example 5.6.
1. If $T = k[f_1, \ldots, f_r]$ is a graded polynomial ring with homogeneous generators $f_1, \ldots, f_r$ of strictly positive degree then $a(T) = \sum_i \deg f_i$.
2. Similarly if $T = k[f_1, \ldots, f_r]/(p_1, \ldots, p_s)$ is a homogeneous normal complete intersection then $a(T) = \sum_i \deg f_i - \sum_j \deg p_j$.

We can now prove a more general version of Proposition 1.4 in the absence of proper semi-invariants.

Proposition 5.7. *Assume that $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$, and $(S\mathfrak{g})^{\mathfrak{g}}$ is finitely generated. Then $a((S\mathfrak{g})^{\mathfrak{g}})$ is defined and is equal to*

$$a((S\mathfrak{g})^{\mathfrak{g}}) = \frac{1}{2}(\dim \mathfrak{g} + i(\mathfrak{g}) - d(\mathfrak{g})) \tag{5.8}$$

Proof. We use the same notations as above. We assume that $\mathfrak{g}$ is non-abelian since otherwise the result is trivial. Since there are no proper semi-invariants, $(S\mathfrak{g})^{\mathfrak{g}}$ is factorial, and hence $a((S\mathfrak{g})^{\mathfrak{g}})$ is defined. Put $a = a((S\mathfrak{g})^{\mathfrak{g}})$. Thus $\omega_X = \mathcal{O}_X(-a)$.

[2] This is a slight abusing of existing terminology as normally the Gorenstein invariant is only defined for Gorenstein rings.

Let $i : \pi^{-1}X_{\mathrm{sm}} \to Y$ be the inclusion map. Applying i_* to (5.7) and using the fact that by Proposition 5.1 $\operatorname{codim}_Y(Y - \pi^{-1}X_{\mathrm{sm}}) \geq 2$ and that everything is reflexive, we obtain an equality

$$\mathcal{O}_Y(-\dim Y - \dim X + d(\mathfrak{g})) = \pi^*\omega_X^{\otimes 2} = \mathcal{O}_Y(-2a)$$

and hence $2a = \dim X + \dim Y - d(\mathfrak{g}) = i(\mathfrak{g}) + \dim \mathfrak{g} - d(\mathfrak{g})$ which yields (5.8). $\square$

This result can also be proved using the method exhibited in [17, Remark 1.6.3] as Proposition 5.1 shows that the set S in loc. cit. (which is U in our terminology) is "big" in the sense of [17].

Example 5.8. We apply Proposition 5.7 to a non-coregular example. Let $\mathfrak{g} = L(6)$ (cfr Example 1.7). Using [8, §89,§93] or the library "ainvar.lib" from Singular [9] we find $(S\mathfrak{g})^{\mathfrak{g}} = k[f_1, f_2, f_3, f_4, f_5,]/(p)$ where

$$\begin{aligned}
f_1 &= v_5^2 - 2v_4v_6\\
f_2 &= v_5^3 - 3v_4v_5v_6 + 3v_3v_6^2\\
f_3 &= v_4^2 - 2v_3v_5 + 2v_2v_6\\
f_4 &= 2v_4^3 + 6v_2v_5^2 + 9v_3^2v_6 - 12v_2v_4v_6 - 6v_3v_4v_5\\
f_5 &= v_6
\end{aligned}$$

and

$$p = f_4f_5^3 - 3f_1f_3f_5^2 + f_1^3 - f_2^2$$

According to Example 1.7 we have $\dim \mathfrak{g} = 6$, $i(\mathfrak{g}) = 4$, $d(\mathfrak{g}) = 0$. The equality (5.8) becomes (taking into account Example 1.7(2))

$$2 + 3 + 2 + 3 + 1 - 6 = 5 = (6 + 4 - 0)/2$$

Remark 5.9. Let g^2 be the fundamental semi-invariant in $S\mathfrak{g}$ (it is a square by Lemma 2.1). If $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$ is a polynomial algebra then it seems that in many cases the irreducible factors of g form a subset of the generators of $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}}$. This is true for Frobenius Lie algebras [2] and also for the examples covered by the methods in [14].

Assume that $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$ and $(S\mathfrak{g})^{\mathfrak{g}} = k[f_1, \dots, f_r]$. If $g = \prod_i f_i^{\epsilon_i}$ then the equality (5.8) becomes

$$\sum_{i=1}^{r}(1 + \epsilon_i)\deg f_i = c(\mathfrak{g})$$

This is similar to a phenomenon observed by Fauquant-Millet and Joseph in [7] that one can sometimes make the inequality (1.1) into an equality by changing the degrees of the f_i in some natural way.

We end this section by proving Proposition 1.6(2)(3).

Proposition 5.10. *Assume that* $(S\mathfrak{g})^{\mathfrak{g}}_{\mathrm{si}} = (S\mathfrak{g})^{\mathfrak{g}}$*,* $\mathfrak{g}$ *is not abelian and* $\mathfrak{g}$ *is coregular. Then* $\operatorname{codim}_{\mathfrak{g}^*}(\mathfrak{g}^* - \mathfrak{g}^*_{reg}) \leq 3$*. If* $\operatorname{codim}_{\mathfrak{g}^*}(\mathfrak{g}^* - \mathfrak{g}^*_{reg}) = 3$ *then* $\mathfrak{g}^* - \mathfrak{g}^*_{reg}$ *is purely of codimension three and is precisely equal to the non-smooth locus of* π*.*

Proof. Since $X = X_{\mathrm{sm}}$ (5.6) yields a complex of vector bundles on Y

$$0 \to \pi^*\Omega_X(1) \xrightarrow{d\pi^*} \mathcal{O}_Y \otimes_k \mathfrak{g} \xrightarrow{\rho} (\mathcal{O}_Y \otimes \mathfrak{g}^*)(1) \xrightarrow{d\pi} \pi^*T_X \to 0 \tag{5.9}$$

which is exact in π^*T_X at the smooth locus of π (denoted by U above). Furthermore the locus in U where it is exact in $(\mathcal{O}_Y \otimes \mathfrak{g}^*)(1)$ is $W \overset{\text{Prop. 5.2}}{=} Y - Y' = \mathfrak{g}^*_{\text{reg}}$.

Assume $\operatorname{codim}_{\mathfrak{g}^*}(\mathfrak{g}^* - \mathfrak{g}^*_{\text{reg}}) \geq 4$. Hence the locus where (5.9) is not exact has codimension ≥ 4. An easy depth computation yields that (5.9) is exact. Thus $\mathfrak{g}^* = \mathfrak{g}^*_{\text{reg}}$ and hence $0 \in \mathfrak{g}^*$ is regular. This is only possible if $\mathfrak{g}$ is abelian, which we had excluded.

Now assume $\operatorname{codim}_{\mathfrak{g}^*}(\mathfrak{g}^* - \mathfrak{g}^*_{\text{reg}}) = 3$. Then by a similar depth computation one finds that (5.9) is exact, except in π^*T_X. Thus in particular $W = U$, or in other words $\mathfrak{g}^* - \mathfrak{g}^*_{\text{reg}}$ coincides with the non-smooth locus of π. We find that $\operatorname{coker} d\pi$ is a Cohen-Macaulay module supported on $\mathfrak{g}^* - \mathfrak{g}^*_{\text{reg}}$. Since Cohen-Macaulay modules have pure support we are done. □

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